Dr. Craig thought quickly, spoke rapidly, and wrote with great celerity. It is true that some of his lectures on differential equations, on hydrodynamics, and on the theory of functions were very advanced and very difficult; and those lectures could be followed with profit only by the maturest of his students.

There is published by the D. Van Nostrand Company his *On the Motion of a Solid in a Fluid.* His most elaborate work *Linear Differential Equations* is published by John Wiley & Sons.

In a private letter, Professor Coffin of Lafayette College says: "His was a wonderfully strong mind."

Likewise, Prof. Simon Newcomb says: "As a student Dr. Craig had a remarkable power of rapid comprehension of mathematical treatises."

Likewise, President Gilman says: "Dr. Craig was a man of unusual aptitudes and of great promise, who began a brilliant career with us; but he suffered long from ill health—and of late years, he was the victim of insomnia."

"Likewise, his father—Mr. Alexander Craig, says: "My son, Thomas Craig, was married to Miss Louise Alvord, of Washington, D. C., May 4, 1880. She is the daughter of the late General Benjamin Alvord of the United States Army"

A Baltimore paper says: "Dr. Thomas Craig, the Professor of Pure Mathematics in the Johns Hopkins University, died suddenly of heart failure, May 8, 1900, at his residence, 1822 St. Paul Street. He had been complaining of feeling unwell for several weeks, although he was at the University as usual that morning. He returned home about noon, and went to his room for a rest before dinner. A member of the family went to call him for dinner and found him dead. Coroner Saunders gave a verdict of death from heart failure."

Dr. Craig is survived by his father and a sister, Miss Margaret Craig, both of Pittston, Pennsylvania; and, also, by his widow and two daughters, Miss Alisa Craig and Miss Ethel Craig, who reside in Baltimore. Says his father: "He was buried at Pittson, Pennsylvania, by the side of his mother whom he always loved so well."

In his Annual Report of the University for 1900 President Gilman writes: "The death of Professor Craig occurred on the 8th of May, 1900, after a long period of declining powers. Those who knew Dr. Craig only in his declining years, need to be told of the enthusiasm, the diligence, and the learning which for a long period were his distinguishing characteristics. He was one of a company of bright young mathematicians who came to the University in its first year, attracted by the brilliant reputation of Professor Sylvester. He showed at once extraordinary powers of acquisition, as well as great ability in the treatment of certain subjects in the domain of higher mathematics. In addition to his contributions to mathematical journals, he published, in 1879, two manuals on the elements of the mathematical theory of fluid motion, and, in 1889, the first volume of a treatise on linear differential equations, a continuation of which was not completed at the time of his death."

In the *American Journal of Mathematics* for October, 1900, Prof. Simon

Newcomb writes as follows: "Thomas Craig, the former editor of this journal, and Professor of Pure Mathematics in the Johns Hopkins University, died May 8, 1900, in his forty-fifth year. His connection with the *American Journal of Mathematics*, as editor or associate editor, continued through the greater part of its existence, being severed at the end of 1898, when failing health compelled him to retire from the editorship. Craig was connected with the Johns Hopkins University from its foundation. He was attracted thither by the desire to pursue mathematical studies under the guidance of Sylvester. From the beginning he showed an extraordinary development of the faculty of acquisition, being able to master, almost without effort, the writings of any of the great geometers to which he was attracted. The productive faculty was developed more slowly.

He was naturally among the earliest Doctors of the University, and the first, or one of the first, to graduate in mathematics. His earliest publications were two small books on hydrodynamics, and a work on projections, prepared for the U. S. Coast Survey, with which he was associated for a short period after his graduation. His most elaborate separate work was a treatise on Linear Differential Equations, embodying the course of instruction on that subject which he gave to the students of the University. A work on higher geometry, on which he was engaged, but, so far as the writer is aware, on which he had made little progress, were left unfinished at the time of his death.

He was also a frequent contributor to the pages of this journal. Among the contributions worthy of especial mention were his various papers on Theta functions, in the fifth and sixth volumes, and a memoir on Linear Differential Equations whose fundamental integrals are the successive derivatives of the same function, in the eighth volume.

During his editorship he devoted himself with great energy to the interests of the *Journal*. The principal object of at least one of his visits abroad was to interest European geometers in it. He recognized and admired the genius of Poincaré; and two elaborate memoirs by the latter, which appeared in the seventh and eighth volumes, were believed to have been sent to the *Journal* on Craig's personal solicitation.

As an expounder of mathematical subjects to advanced students, Craig's abilities were of a high order. His lectures were well prepared, and he spoke with rapidity, clearness and force. It may well be that only the best students were able to keep up with him, but these profited in a high degree from his expositions and entertained a permanent appreciation of his efforts for their development. Concentrating his interests almost entirely on his family and his students, rarely taking a long rest, he mingled little with men, especially in his later years, when his activities were greatly restricted by failing health."

At a meeting of the (Johns Hopkins) Board of University Studies, held May 23, 1900, the following minute was unanimously adopted:

"The members of the Board of University Studies of the Johns Hopkins University desire to express their sorrow at the death of their friend and colleague, Prof. Thomas Craig, who, as student and teacher of mathematics, had

been connected with the University for nearly the entire period of its existence. One of the brilliant young men whom Professor Sylvester attracted to the University in its early days, he won straightway the favorable notice of that eminent man for the enthusiasm and intellectual acumen with which he entered upon the study of advanced mathematics, then almost an unknown science in this country; and this fortunate combination of interest, energy, and ability characterized his entire career. At the time of his death he was occupied in the preparation of a treatise on the Theory of Surfaces. Undoubtedly the intense ardor with which he engaged in this work contributed in large measure to that impairment of the nervous system from which he had recently suffered. Professor Craig possessed great power of research, and wrote much for various mathematical journals. For many years he was editor of the *American Journal of Mathematics*, and it is largely due to his zeal and able direction that that journal continues to hold its high rank in the mathematical world. Professor Craig occupied a place in the very front rank of American mathematicians. His scientific ideals were the highest, and as teacher, editor, and investigator, he brought to his work a high degree of originality, and an intellectual ardor which was a source of inspiration to all with whom he was closely associated."

ON SYSTEMS OF ISOTHERMAL CURVES.*

By PROFESSOR L. E. DICKSON.

1. The object of this paper is to give an elementary geometrical definition of a system of isothermal curves in the plane. The definition is readily extended to families of curves on any algebraic surface. For simplicity of expression, the definition is given in connection with the two families of curves which are to be discussed at length; the general definition will then be apparent. The usual methods of treating the subject are indicated in §§ 4-5.

2. The concentric circles about a point O have as orthogonal trajectories the straight lines through O. Select two of these lines, OP and OQ, and designate by q the number of radians in the angle POQ. On the circle about O with radius $OP=r$, the arc $PQ=qr$. On the line OP measure off from P the length $PT=qr$. [In Fig. 1, it is taken to the right of P]. On the circle about O with radius $OT=r+qr$, the arc TS equals $qr+q^2r$. Hence PQ, PT and QS are each of length qr, while the limit of $TS \div PT$ as q approaches zero is unity.

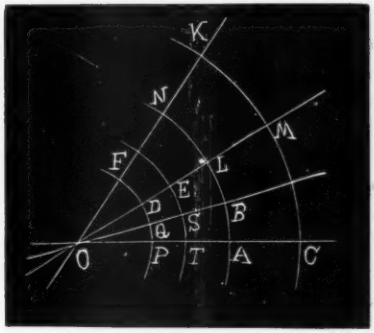


Fig. 1.

*Read before the American Association for the Advancement of Science.

Let L be an arbitrary point in the plane, but distinct from O. Denote by A and B the intersections of OP and OQ, respectively, with the circle about O with radius OL. Denote by D and E the intersections of OL with the circles through P and T. We define a curvilinear quadrilateral, one of whose vertices is L, as follows: Set $OA=a$, and denote by d the number of radians in the angle POD. Then arc $AB=qa$, $DE=qr$. On OP take $AC=AB=qa$; on the circle PQ take arc $DF=DE=qr$. The lines OD, OF and the circles about O with radii OA, OC intersect in the curvilinear quadrilateral $LNKM$. Its sides have the following lengths:

$$LM=qa, \quad LN=qa, \quad NK=qa, \quad MK=q(a+qa).$$

Hence, as q approaches zero, the limits of the ratios of the sides are all unity. The angles of the quadrilateral are always $\pi/2$. In the vicinity of an arbitrarily chosen point L, we obtain by this construction a network of rectangular curvilinear quadrilaterals, which tend to become squares as q approaches zero. The family of concentric circles and their orthogonal trajectories are therefore said to form a system of isothermal curves.

Instead of the particular "base lines" OP, OQ, circle PQ, and circle TS, we may employ any other pair of straight lines through O and pair of circles about O, such that the limits of the ratios of the quadrilateral formed are all unity.

3. Consider the family of circles $x^2+(y-r)^2=r^2$ tangent to the x-axis at the origin. In polar coördinates, their equations are

$$(1) \qquad \rho=2r\sin\theta \quad (r=\text{parameter}).$$

Their orthogonal trajectories are the circles tangent to the y-axis at the origin; their equations are

$$(2) \quad \rho=2R\cos\theta \quad (R=\text{parameter}).$$

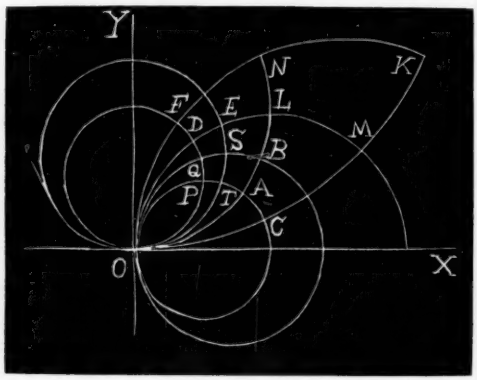


Fig. 2.

We are to show that these curves form an isothermal system, according to the definition next given. Employing as base lines certain of the circles, PQ, TS, PT, and QS, we define a curvilinear quadrilateral $LNKM$, one of whose vertices L is an arbitrarily chosen point distinct from O, by making arc AB=arc AC, arc DF=arc DE, thereby determining the circles $OCMK$ and $OFNK$. If $PT=PQ$, then $LNKM$ tends to become a square as PQ approaches zero.

In proof, we consider the circles of unit radius

$$OPQ : \rho=2\sin\theta ; \quad OPT : \rho=2\cos\theta.$$

Their second point of intersection P has the coördinates

$$P : (\rho, \theta)\equiv(\sqrt{2}, \pi/4).$$

Let the point Q be determined on the circle OPQ so that the angle POQ shall contain q radians. Then arc $PQ=2q$. Take arc $PT=2q$ on the circle OPT. Since angle $XOQ=\frac{1}{4}\pi+q$, the length of OQ is

$$\rho=2\sin(\tfrac{1}{4}\pi+q)=\sqrt{2}(\cos q+\sin q)=\sqrt{2}(1-\frac{q^2}{2!}+\frac{q^4}{4!}-\ldots.+q-\frac{q^3}{3!}+\frac{q^5}{5!}-..).$$

Hence the coördinates of Q are

$$Q : \rho=\sqrt{2}(1+q+\ldots.). \quad \theta=\tfrac{1}{4}\pi+q,$$

where, as henceforth, the terms indicated by dots contain the factor q^2. By a like proof, or by symmetry, the coördinates of T are

$$T : \rho=\sqrt{2}(1+q+ \ldots), \quad \theta=\tfrac{1}{4}\pi - q.$$

In order that a circle (2) shall contain Q, we find that

$$R=\frac{\sqrt{2}(1+q+\ldots.)}{2\cos(\frac{1}{4}\pi+q)}=\frac{1+q+\ldots.}{1-q+\ldots.}=1+2q+\ldots.$$

Hence the equation of the circle in question is

$$OQSB : \rho=2\cos\theta(1+2q+\ldots).$$

Analogously, or by symmetry, the circle (1) through T is

$$OTSE : \rho=2\sin\theta(1+2q+\ldots.).$$

Consider an arbitrary point L. It may be determined as the intersection of the circle $OABL$ of set (1) with the circle $ODEL$ of set (2). These circles are determined by the points A and D, respectively. Let them have the coördinates

$$A : (\rho, \theta)\equiv(2\cos\alpha, \alpha); \quad D : (\rho, \theta)\equiv(2\sin\beta, \beta).$$

Then the circles $OABL$ and $ODEL$ have the respective equations

$$\rho=2\cot\alpha.\sin\theta, \quad \rho=2\tan\beta.\cos\theta.$$

The former intersects the circle $OQSB$ in the point B for which

$$\tan\theta_B = \tan\alpha.(1+2q+\dots).$$

Denoting by σ the angle AOB, we have $\theta_B = \alpha+\sigma$, so that

$$\tan\alpha.(1+2q+\dots) = \frac{\tan\alpha+\sigma+\frac{1}{3}\sigma^3+\dots}{1-\tan\alpha(\sigma+\frac{1}{3}\sigma^3+\dots)} = \tan\alpha+\sigma+\sigma\tan^2\alpha+\sigma^2(\ ).$$

Hence $\sigma = q\sin 2\alpha+\dots$. Since the radius of $OABL$ is $\cot\alpha$, we get

$$\text{arc}AC \equiv \text{arc}AB = 2\sigma\cot\alpha = 4q\cos^2\alpha+\dots.$$

Hence the point C has the coördinates

$$\rho = 2\cos\alpha+4q\sin\alpha\cos^2\alpha+\dots., \quad \theta = \alpha-2q\cos^2\alpha+\dots.$$

In order that circle a (1) shall pass through C, we must have

$$r = \frac{\cos\alpha+2q\sin\alpha\cos^2\alpha+\dots.}{\sin(\alpha-2q\cos^2\alpha+\dots.)} = \frac{\cos\alpha(1+2q\sin\alpha\cos\alpha+\ \dots)}{\sin\alpha-2q\cos^3\alpha+\dots.}$$

$$= \frac{\cos\alpha}{\sin\alpha}[1+2q(\sin\alpha\cos\alpha+\frac{\cos^3\alpha}{\sin\alpha})+\dots.] = \cot\alpha(1+2q\cot\alpha+\dots.).$$

The equation to the circle $OCMK$ is therefore

$$\rho = 2\sin\theta(\cot\alpha+2q\cot^2\alpha+\dots.).$$

For the intersection E of the circles $ODEL$ and $OTSE$, we have

$$\tan\theta_E = \tan\beta.(1-2q+\dots.), \quad \theta_E = \beta-q\sin 2\beta+\dots$$

$\therefore$ $\text{arc}DF \equiv \text{arc}DE = 4q\sin^2\beta+\ \dots$
Hence the point F has the coördinates

$$\rho = 2\sin\beta+4q\cos\beta\sin^2\beta+\dots., \quad \theta = \beta+2q\sin^2\beta+\dots.$$

The equation to the circle $OFNK$ of family (2) is therefore

$$\rho = 2\cos\theta(\tan\beta+2q\tan^2\beta+\dots.).$$

We may now determine the coördinates of L, M, N from the equations of the circles $OABL$, $ODEL$, $OCMK$, and $OFNK$:

$$\tan\theta_L = \tan\alpha\tan\beta, \quad \rho_L = 2(\tan^2\alpha+\cot^2\beta)^{-\frac{1}{2}},$$

$$\tan\theta_M = \tan\alpha\tan\beta - 2q\tan\beta + \dots,$$
$$\tan\theta_N = \tan\alpha\tan\beta + 2q\tan\alpha\tan^2\beta + \dots.$$

Denoting by τ the angle MOL, then $\theta_M = \theta_L - \tau$, whence

$$\tan\alpha\tan\beta - 2q\tan\beta + \dots = \frac{\tan\theta_L - \tau - \frac{1}{3}\tau^3 \dots}{1 + \tan\theta_L(\tau + \frac{1}{3}\tau^3 + \dots)}$$

$$= \tan\alpha\tan\beta - \tau(1 + \tan^2\alpha\tan^2\beta) + \tau^2(\ ).$$

Since the radius of $ODEL$ is $\tan\beta$, we have

$$\text{arc}ML = 2\tau\tan\beta = \frac{4q\tan^2\beta}{1 + \tan^2\alpha\tan^2\beta} + \dots.$$

Determining the angle $LON \equiv \theta_N - \theta_L$, we find similarly that

$$\text{arc}LN = 2\cot\alpha(LON) = \frac{4q\tan^2\beta}{1 + \tan^2\alpha\tan^2\beta} + \dots.$$

Hence

$$\lim_{q=0} \frac{\text{arc}ML}{\text{arc}LN} = 1,$$

so that the families of circles (1) and (2) form an isothermal system. Also

$$\text{arc}LN = q\rho_L^2 + \dots.$$

$$\lim_{q=0} \frac{\text{arc}LN}{\text{arc}PQ} = \frac{2}{\tan^2\alpha + \cot^2\beta} = \tfrac{1}{2}\rho_L^2.$$

A curviliner quadrilateral, whose angles are all $\pi/2$ and the limits of the ratios of whose sides are all unity as q approaches zero, may be designated as an "infinitesimal square." We may state our result in the following terms:

The two families of circles (1) *and* (2) *form a network of infinitesimal squares whose sides are proportional to the squares of the distances from the origin to their nearest corner points.*

4. All systems of isothermal curves may be obtained by function-theory as follows: Set

$$X + iY = \theta(x + iy), \quad X - iY = \bar{\theta}(x - iy).$$

$$\therefore X = \tfrac{1}{2}[\theta(x+iy) + \bar{\theta}(x-iy)] \equiv U(x, y), \quad Y \equiv \frac{-i}{2}(\theta - \bar{\theta}) \equiv V(x, y).$$

The lines $X = a$, $Y = b$ form an isothermal system. Under the transformation $X = U(x, y)$, $Y = V(x, y)$, the straight lines correspond to the curves $U(x, y) = a$, $V(x, y) = b$. The point (a, b) corresponds to the intersection P_1 of

$U=a$, $V=b$; the point $(a, b+q)$ corresponds to the intersection P_2 of $U=a$, $V=b+q$; the point $(a+q, b)$ corresponds to the intersection P_3 of $U=a+q$, $V=b$. It may be proved that angle $P_3P_1P_2=\frac{1}{2}\pi$, and that

$$\lim_{q=0} \frac{P_1P_2}{P_1P_3}=1.$$

To obtain the correspondence in which the sense of the angles is reversed, we set $X-iY=\theta(x+iy)$.

5. Isothermal systems may be treated from the standpoint of transformation-groups.* For the case of the orthogonal circles (§3), the system may be derived from the system of lines parallel to the axes by the familiar transformation through reciprocal radii vectors.† The latter transforms any isothermal system into an isothermal system since it leaves invariant the partial differential equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,$$

satisfied by the function $U\equiv U(x, y)$ of §4.

The University of Chicago, February, 1901.

ATMOSPHERIC REFRACTION.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The object of this article is not to set forth a new theory but simply to review the old. Let x=radius vector from center of the earth to any point in the path of the ray, φ=the angle between the ray and its normal, μ=the index of refraction, a, θ, μ_0=the values of x, φ, μ at the earth's surface, r=the atmospheric refraction.

Then $\mu x\sin\varphi=\mu_0 a\sin\theta$ . . . . (1).

Let φ'=the angle a consecutive element of the ray's path makes with the normal. Then $\varphi-\varphi'=dr$. . By the laws of refraction

$$\mu\sin\varphi=(\mu+d\mu)\sin(\varphi-dr)=(\mu+d\mu)(\sin\varphi-dr\cos\varphi)$$

since $\sin dr=dr$ and $\cos dr=1$.

$\therefore$ $d\mu\sin\varphi=\mu dr\cos\varphi$.

$\therefore$ $dr=\frac{1}{\mu}\tan\varphi d\mu$, and $r=\int\frac{\tan\varphi d\mu}{\mu}$.

---

*Lie-Scheffers, *Differentialgleichungen*, pp. 156-157.

†Lie-Scheffers, *Geometrie der Beruhrungstransformationen*, pp. 6-9.

According to Simpson the law of decrease of density of the atmosphere is such that some power of the refractive index varies inversely as the distance from the center of the earth.

$$\therefore \left(\frac{\mu}{\mu_0}\right)^{n+1} = \frac{a}{x}.$$

This in (1) gives $\sin\varphi = \left(\frac{\mu}{\mu_0}\right)^n \sin\theta$.

$$\therefore \tan\varphi = \frac{\mu^n \sin\theta}{\sqrt{(\mu_0^{2n} - \mu^{2n}\sin^2\theta)}}.$$

$$\therefore r = \int_1^{\mu_0} \frac{\mu^{n-1}\sin\theta\, d\mu}{\sqrt{(\mu_0^{2n} - \mu^{2n}\sin^2\theta)}} = \frac{1}{n}\left[\theta - \sin^{-1}\left(\frac{\sin\theta}{\mu_0^{\,n}}\right)\right].$$

$$\therefore \sin\theta = \mu_0^{\,n}\sin(\theta - nr) \ldots . (2).$$

Since nr is small, we write $\cos nr = 1$, $\sin nr = nr$.

$$\therefore \sin\theta = \mu_0^{\,n}(\sin\theta - nr\cos\theta).$$

$$\therefore r = \frac{(\mu_0^{\,n} - 1)\tan\theta}{n\mu_0^{\,n}} \ldots . (3).$$

From (2) $$\frac{\sin\theta}{\sin(\theta - nr)} = \mu_0^{\,n}.$$

$$\therefore \frac{\sin\theta - \sin(\theta - nr)}{\sin\theta + \sin(\theta - nr)} = \frac{\mu_0^{\,n} - 1}{\mu_0^{\,n} + 1}.$$

$$\therefore \tan\frac{nr}{2} = \frac{\mu_0^{\,n} - 1}{\mu_0^{\,n} + 1}\tan\left(\theta - \frac{nr}{2}\right) \ldots . (4).$$

It has been demonstrated by the experiments of Biot and Arago that $\mu_0^2 - 1 = 4k\rho$, where ρ is the density of the atmosphere at the earth's surface and k is a constant so small that k^2 can be neglected.

Hence $\mu_0^{\,n} = (1 + 4k\rho)^{\frac{1}{2}n} = 1 + 2nk\rho \ldots . (5)$.

This in (4) gives

$$\tan\frac{nr}{2} = \frac{nk\rho}{1 + nk\rho}\tan\left(\theta - \frac{nr}{2}\right). \quad \text{Now } \tan\frac{nr}{2} = \frac{nr}{2}.$$

$$\therefore r = \frac{2k\rho}{1 + nk\rho}\left(\frac{\tan\theta - \frac{1}{2}(nr)}{1 + \frac{1}{2}(nr)\tan\theta}\right).$$

$$\therefore r^2 + \frac{2r(1 + nk\rho)}{n\tan\theta} = \frac{4k\rho}{n}.$$

$$r=\frac{\sqrt{(4nk\rho\tan^2\theta+1+2nk\rho)}-(1+nk\rho)}{n\tan\theta}\ldots.(6).$$

(5) in (3) gives $r=\dfrac{2k\rho\tan\theta}{1+2nk\rho}\ldots.(7).$

We must divide (6) and (7) by sin1″ to reduce to seconds.

$$r=\frac{\sqrt{(4nk\rho\tan^2\theta+1+2nk\rho)}-(1+nk\rho)}{n\tan\theta\sin1''}\ldots.(8).$$

$$r=\frac{2k\rho\tan\theta}{(1+2nk\rho)\sin1''}\ldots.(9).$$

When $\theta=\frac{1}{2}\pi$, (8) becomes $r=\sqrt{\dfrac{4k\rho}{n}}\cdot\dfrac{1}{\sin1''}\ldots.(10).$

θ is the zenith distance. For 0° *C.* and 760 *MM.* experiment shows $4k=$ .000588768, and $\rho=1$.

From (10), $n=\dfrac{4k\rho}{(r\sin1'')^2}$, but $r=34'\ 30''=2070'$ when $\theta=\frac{1}{2}\pi$. This is the mean value for many observations.

$$\therefore\ n=\frac{.000588768}{(2070\times.000004848)^2}=5.8463.$$

n and $k\rho$ in (9) and (8) give $r=60.6\tan\theta\ldots.(11).$

$$r=\frac{\sqrt{(1.00172106+.00344211\tan^2\theta)}-1.00086053}{.00002834\tan\theta}.$$

$$\therefore\ r=\frac{35316.17[\sqrt{(1+.003436\tan^2\theta)}-1]}{\tan\theta}\ldots.(12).$$

$$=60.67\tan\theta-.0522\tan^3\theta\ldots.(13).$$

(12) is applicable for all values of θ. (11) and (13) are accurate enough for values of θ up to 70° only.

Let g=gravity at surface, p=pressure, and δ=density of air at a distance x from center of earth. n is, also, found as follows:

$$dp=-\frac{a^2g\delta dx}{x^2}=ag\delta d\ \frac{a}{x}.$$

Now $\dfrac{a}{x}=\left(\dfrac{\mu}{\mu_0}\right)^{n+1}=\left(\dfrac{1+4k\delta}{1+4k\rho}\right)^{\frac{1}{4}(n+1)}=1-2(n+1)k(\rho-\delta).$

$\therefore\ d(a/x)=2k(n+1)d\delta.$

$\therefore\ dp=2k(n+1)ag\delta d\delta,\ p=agk(n+1)\delta^2.$

Let l=height of a homogenous atmosphere of density ρ exerting a pressure p_0.

$\therefore\ p_0=g\rho l.$

$\therefore\ \frac{p}{p_0}=(n+1)\ \frac{a}{l}\cdot\frac{k\delta^2}{\rho}$, when $p=p_0,\ \delta=\rho.$

$\therefore\ 1=(n+1)\ \frac{a}{l}\ k\rho$, or $n=\frac{l/a}{k\rho}-1.$

a=6366738 meters, $k\rho$=.000147192.

l=.76 (as many times as mercury is as heavy as air)=7993.15 meters.

$\therefore\ l/a$=.00125545.

$\therefore\ n=\frac{.00125545}{.000147192}-1$=7.5, a value of n too large.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

146. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If the driving-wheels of Locomotive No. 200 on the Pennsylvania Railroad, m=7 feet in diameter, turn n=20 times in p=3 seconds, and lose r=12% of their forward motion by slipping on the smooth steel rails, at what rate per hour is the locomotive moving over the rails?

Solution by W. P. WEBBER, Mississippi Normal College, Houston, Miss.

$m\pi$=circumference of wheel, $mn\pi$=number of times the circumference is applied to the rail, and is therefore the distance the engine travels without slipping in p seconds.

Hence, $\frac{mn\pi}{p}$=distance engine travels in 1 second without slipping, and $\frac{mn(100-r)\pi}{100p}$=actual distance engine travels in 1 second, since it slips back r% of the distance it travels.

Substituting numbers for the letters, we have for the distance the engine travels in 1 second, $\frac{7\times20(100-12)\pi}{100\times3}$ feet.

Hence, the rate of the engine in miles per hour is

$$\frac{7\times 20\times 88\times \pi\times 3600}{100\times 3\times 5280}=28\pi \text{ miles.}$$

ALGEBRA.

123. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}}=2\sqrt[4]{\frac{1-a^2}{(1+a)^2}}$, and $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}$.

[*Haddon.*]

Solution by JOHN A. VAN GROOS, Fellow of Mathematics, University of Oregon, Eugene; Ore.; R.L.MOORE, Student in University of Texas, 2206 San Marcos Street, Austin, Tex.; and G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1). Multiply $\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}}=2\sqrt[4]{\frac{1-a^2}{(1+a)^2}}=2\sqrt[4]{\frac{1-a}{1+a}}$ through by

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}. \quad \therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-2\left(\frac{1-a}{1+a}\right)^{\frac{1}{4}}\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\left(\frac{1-a}{1+a}\right)^{\frac{1}{4}}=0.$$

$$\therefore \left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-\left(\frac{1-a}{1+a}\right)\right]^2=0. \quad \therefore \frac{1+x}{1-x}=\frac{1-a}{1+a}.$$

$\therefore x=-a.$

(2). $\sqrt{(a^2-x^2)}=a^2\sqrt{(1-x^2)}-x\sqrt{(a^2-1)}.$

$$a^2-x^2=a^4-a^4x^2-2a^2x\sqrt{[(1-x^2)(a^2-1)]}+a^2x^2-x^2.$$

$\therefore 2a^2x\sqrt{[(1-x^2)(a^2-1)]}=a^2(1-x^2)(a^2-1).$

$\therefore \{2x-\sqrt{[(1-x^2)(a^2-1)]}\}\sqrt{[(1-x^2)(a^2-1)]}=0.$

$\therefore 4x^2=(1-x^2)(a^2-1)$ or $x^2=1.$

$\therefore x=-1,\ +1,\ \text{or}\ \pm\sqrt{\frac{a^2-1}{3+a^2}}.$

The values $x=-1$ and $\sqrt{\frac{a^2-1}{3+a^2}}$ are the values satisfying the equation as given.

124. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A certain quantity of alcohol diluted with water so that in one liter there are c liters of pure alcohol, is mixed n times successively with p times the quantity of alcohol diluted so that 1 liter contains a liter of pure alcohol. How much pure alcohol does one liter of the nth mixture contain?

Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Total put in.	*Alcohol put in.*	*Total.*
1	c	1
p	ap	$1+p$
$(1+p)p$	$(1+p)ap$	$(1+p)^2$
$(1+p)^2p$	$(1+p)^2ap$	$(1+p)^3$
⋮	⋮	⋮
$(1+p)^np$	$(1+p)^nap$	$(1+p)^{n+1}$

The sum of the quantities in the second column is $c-a+a(1+p)^{n+1}$, and this divided by $(1+p)^{n+1}$ is the answer.

GEOMETRY.

151. Proposed by FRANK A. GIFFIN, Assistant in Mathematics, University of Colorado, Boulder, Col.

A point at a distance of 1 inch, 2 inches, and $2\frac{1}{2}$ inches, respectively, from three corners of a square. Construct the square. Also solve for the general distances a, b, c.

II. Solution by MARCUS BAKER, U. S. Coast Survey, Washington, D. C.

In the June-July number of the MONTHLY was published an *analysis* of this *without construction* as called for. The following is presented as an analysis *and* construction.

This is a variation of Rutherford's problem, which is: Given the distances a, b, c, of any point in the plane of an equilateral triangle, from the vertices of that triangle required to construct it. In the problem here proposed a right angled isosceles triangle replaces the equilateral triangle of Rutherford's problem.

Analysis. The triangle ABC is half of a square whose side is x, *i. e.* it is an isosceles triangle right angled at A. Let P be any point. Join P to A, B, and C by the lines a, b, and c, dividing the original triangle into three triangles. Now imagine each of these three triangles folded over its corresponding side of the original triangle in such wise that P falls at P_a, P_b, P_c. Complete the figure. Then there results a pentagon whose area is obviously twice that of ABC, *i. e.* equals x^2. This pentagon is composed of three triangles, viz:

P_cP_aB,	sides b, b, $b\sqrt{2}$,	area$=\frac{1}{2}b^2$
P_aP_bC,	sides c, c, $c\sqrt{2}$,	area$=\frac{1}{2}c^2$
$P_aP_bP_c$,	sides $b\sqrt{2}$, $c\sqrt{2}$, $2a$,	area$=$

$$\left[\left(\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}+a\right)\left(-\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}+a\right)\left(\frac{b}{\sqrt{2}}-\frac{c}{\sqrt{2}}+a\right)\left(\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}-a\right)\right]^{\frac{1}{2}}$$

Construction. From the above analysis it is obvious that we must first

construct the triangle $P_aP_bP_c$, whose sides are known, and then upon $b_1/2$ and $c_1/2$ construct isosceles right triangles. The vertices of these triangles (at the right angles) and the middle of the side $2c$ are the vertices of the required triangle.

Number of Solutions. In constructing $P_aP_bP_c$ we may take for its sides

$$\begin{array}{lllll} & a_1/2, & b_1/2, & 2c & \ldots.(1). \\ \text{or} & a_1/2, & 2b, & c_1/2 & \ldots(2). \\ \text{or} & 2a & b_1/2, & c_1/2 & \ldots.(3). \end{array}$$

In the special problem before us $a=1$, $b=2$, $c=2\frac{1}{2}$; whence

$$\begin{array}{lll} 1.414, & 2.828, & 5. \quad \ldots.(1). \\ 1.414, & 4. \quad , & 3.535\ldots.(2). \\ 2, \quad , & 2.828, & 3.535\ldots.(3). \end{array}$$

In the first case there is no real solution.

In the second case P falls *without* the triangle, and $x=2\frac{7}{16}$.

In the third case P falls *within* the triangle, and $x=2\frac{3}{16}$.

The values are derived by scaling off from the figures. The construction here given is for the third case, the unit being one centimeter.

154. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

The angle between the edge of a trihedral angle and the bisector of the opposite face angle is less than, equal to, or greater than half the sum of the other two face angles, according as it is itself acute, right, or obtuse.

Solution by the PROPOSER.

Let $S-ABC$ be a trihedral angle and SD the bisector of the face angle ASB.

CASE I. $\angle DSC <$ a right angle. (See Fig. 1.)

To prove $\angle DSC < \frac{1}{2}(\angle ASC + \angle BSC)$.

From C, any point of the edge SC, draw CD perpendicular to SD.

Through D, in the face ASB, draw AB perpendicular to SD. Then, SD is perpendicular to the plane ABC.

Connect S with F and E.

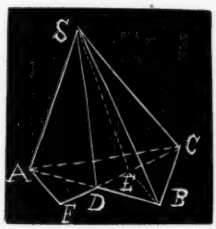


Fig. 1.

Comparing the right triangles AFD and BED, $AD=BD$ (since triangle ASD=triangle BSD), and the vertical angles at D are equal. Hence, the triangles are equal, and $DE=DF$.

It follows that right triangles FSD and ESD are equal, and $\angle FSD = \angle ESD$.

Now, since SD is perpendicular to the plane ABC, the plane DSC is perpendicular to the plane ABC. Therefore BE and AF, which lie in one of these planes and are perpendicular to their intersection, are perpendicular to the other plane, DSC.

Consequently, SE and SF are the projections of SB and SA, respectively, on the plane DSC.

Therefore $\angle ESC < \angle BSC$, and $\angle FSC < \angle ASC$; or $\angle DSC - \angle DSE < \angle BSC$, and $\angle DSC + \angle DSF < \angle ASC$.

Adding, and remembering that $\angle DSE = \angle DSF$, $2\angle DSC < \angle BSC + \angle ASC$, $\angle DSC < \frac{1}{2}(\angle BSC + \angle ASC$.

Case II. $\angle DSC$ = a right angle. (See Fig. 2.)

To prove $\angle DSC = \frac{1}{2}(\angle ASC + \angle BSC)$.

Draw DX parallel to SC, and AB perpendicular to SD.

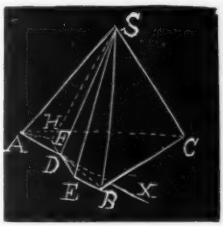


Fig. 2.

Then SD is perpendicular to DX, and, hence, to the plane determined by AB and DX.

This plane intersects the planes of faces BSC and ASC in BE and AF, respectively.

Since SC is parallel to DX, it is parallel to the plane BDX, and, hence, parallel to BE and AF.

Through SD pass a plane perpendicular to SC, intersecting the plane of AB and DX in EF, and the planes of faces BSC and ASC in SE and SF, respectively. Since CS is perpendicular to the plane FSE, so are BE and AF.

Hence, $\angle$'s BES, BED, AFS, and AFD are right angles.

Right triangles DAF and DBE are equal, since $AD = BD$ (from equality of right triangles ASD and BSD), and the vertical angles at D are equal.

Therefore $BE = AF$. Hence, since $SB = SA$, right triangles SEB and SFA are equal, and $\angle ASF (= \angle ASH) = \angle BSE$.

Now, since $\angle$'s CSD, CSE, and CSH are right angles, $\angle CSD = \frac{1}{2}(\angle CSE + \angle SCH)$, $= \frac{1}{2}(\angle CSB + \angle BSE + \angle CSA - \angle ASH)$
$= \frac{1}{2}(\angle CSB + \angle CSA)$.

Case III. $\angle DSC$ > a right angle. (See Fig. 3.)

To prove $\angle DSC > \frac{1}{2}(\angle CSA + \angle CSB)$.

Produce AS, BS, and DS, forming another trihedral angle $S - A'B'C$.

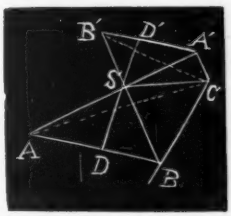


Fig. 3.

By Case I, $\angle CSD' < \frac{1}{2}(CSA'(\angle CSA' + \angle CSB')$, or $180° - \angle CSD < \frac{1}{2}(180° - \angle CSA + 180° - \angle CSB)$, from which $-\angle CSD < -\frac{1}{2}(\angle CSA + \angle CSB)$.

Therefore, $\angle CSD > \frac{1}{2}(\angle CSA + \angle CSB)$.

Also solved by *H. C. WHITAKER*.

155. Proposed by J. C. NAGLE, M.A., C.E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Texas.

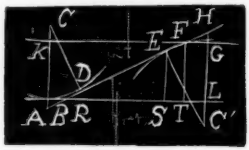
A special case of the following problem was sent me some time ago by an ex-member of one of my engineering classes, as occurring on the Southern Pacific Ry., near Devil's River:

Two straight tracks, p feet between centers, are to be united by a cross-over composed of two curves of radius R, and a length L of intervening tangent. Required the central angles and the distance between tangent points, measured along main track. In the special case referred to p was 62 feet, L 100 feet with 9° 30′ curves.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\tan OCD=\tan ECG=m$, $OC=CD=CG=CE=R$, $DE=L$, $KO=LG=p$. The equation to the line AH is $y=mx-b$, where O is origin and $b=OA$.

$CA=CD\sec OCD$. $\therefore R+b=R\sqrt{(1+m^2)}$.



$\therefore y=mx+R-R\sqrt{(1+m^2)}$.

When $y=0$, $x=R[\sqrt{(1+m^2)}-1]/m$,

When $y=p$, $x=[p+R\sqrt{(1+m^2)}-R]/m=x_1$.

$\therefore OB=BD=EF=FG=R[\sqrt{(1+m^2)}-1]/m$.

Since $BF=\sqrt{[BT^2+TF^2)}=\sqrt{[(x_1-x)^2+p^2]}$,

$\therefore L+2R[\sqrt{(1+m^2)}-1]/m=\sqrt{[(p/m)^2+p^2]}=(p/m)\sqrt{(1+m^2)}$.

$\therefore [(2R-p)^2-L^2]m^2+4RLm=4Rp-p^2$. This determines m.

$OL=OT+TL=x+x_1=[p+2R\sqrt{(1+m)^2}-2R]/m$.

$BR=BD\cos OCD=x/\sqrt{(1+m^2)}=R[\sqrt{(1+m^2)}-1]/[m\sqrt{(1+m^2)}]$.

$RS=BT-2BR=p/m-2R[\sqrt{(1+m^2)}-1]/[m\sqrt{(1+m^2)}]$.

$p=62$, $L=100$, $R=604$.

$\therefore 325829m^2+60400m=36487$ or $m=.254559$.

$\therefore \angle OCD=14^\circ\ 16'\ 52.6''=$central angles.

$OL=394.9$ feet, $RS=96.9$ feet.

CALCULUS.

112. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A sphere of radius r is pierced by a cylinder radius $\frac{1}{2}r$ so that the cylinder just grazes the center of the sphere. Find volume removed; the lateral surface and the spherical surface removed.

Solution by L. C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and the PROPOSER.

Taking the center of the sphere for coördinates, we have for the equations of sphere and cylinder, respectively,

$$x^2+y^2+z^2=r^2,\quad y^2=rx-x^2.$$

Therefore volume removed, $V=4\int_0^r dx\int_0^{\sqrt{(rx-x^2)}} z\,dy$

$$=2\int_0^r\left[(r^2-x^2)\sin^{-1}\sqrt{\frac{x}{r+x}}\,dx+\sqrt{(rx)}(r-x)dx\right]=\frac{3\pi-4}{9}.2r^3.$$

The lateral surface $S=2\int z\partial s$, s being an arc of the base circle of the cylinder. From $y^2=rx-x^2$,

$$\frac{dy}{dx}=\frac{r-2x}{2\sqrt{(rx-x^2)}},\qquad \frac{dz^2}{dx^2}=1+\left(\frac{dy}{dx}\right)^2=\frac{r^2}{4(rx-x^2)},\qquad z^2=r^2-x^2-y^2=r^2-rx.$$

$$\therefore\ L=2r\sqrt{r}\int_0^r\frac{dx}{\sqrt{x}}=4r^2.$$

The spherical surface removed, $S=4\int_0^r dx\int_0^{\sqrt{(rx-x^2)}}dy\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}$

$$=4r\int_0^r dx\int_0^{\sqrt{(rx-x^2)}}\frac{dy}{\sqrt{(r^2-x^2-y^2)}}=4r\int_0^r\sin^{-1}\sqrt{\frac{x}{r+x}}.dx=(\pi-2)2r^2.$$

NOTE. The results of Problem 6 in Byerly's *Integral Calculus*, page 282, 2nd Ed. (1898) are correct, if b represents the diameter of the cylindrical hole instead of the radius.

Also solved by *H. C. WHITAKER*.

113. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

At what rate per unit of time are the roots of the equation $(x^2+px+q=0)$ changing, if $p=mq$ and q varies uniformly at the rate of 1/12 per unit of time, when $p=12$ and m remains constant?

Solution by H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

The roots of the given equation are

$$x=\tfrac{1}{2}[-mq\pm\sqrt{(m^2q^2-4q)}],$$
$$dx=\tfrac{1}{2}[-m\pm(m^2q-2)(m^2q^2-4q)^{-\frac{1}{2}}]dq.$$

When $q=12\div m$ and $dq=1\frac{1}{2}$,

$$dx=\tfrac{1}{24}\left[-m\pm\frac{(6m-1)\sqrt{m}}{2\sqrt{(9m-3)}}\right].$$

Also solved by *WILLIAM HOOVER*.

114. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

If two concentric ellipses have equal axes inclined at an angle ω, their common area is

$$A=2ab\tan^{-1}\left(\frac{2ab}{(a^2-b^2)\sin\omega}\right).$$

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and J. B. GREGG, Senecaville, Ohio.

Let AB be the semi-major axis of one ellipse, and AB' that of the other, MAC and NAD being common diameters. F is the intersection of AB with the second ellipse, and E and G are the intersections of major axis of the second ellipse with the first.

Let angle $B'AB=\omega$. It can be easily shown that angle $CAF=\frac{1}{2}\omega$, and angle $DAF=$angle $DAG=\frac{1}{2}\pi-\frac{1}{2}\omega$.

The polar equation of an ellipse referred to the center as pole is

$$r^2=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

Since section $EAC=CAF$, and $DAF=DAG$; we have for $ECFA$ integral

$$a^2b^2\int_{\frac{1}{2}\omega}^{\omega}\frac{d\theta}{a^2\sin^2\theta+b^2\cos^2\theta}=ab\left[\tan^{-1}\left(\frac{a}{b}\tan\omega\right)-\tan^{-1}\left(\frac{a}{b}\tan\tfrac{1}{2}\omega\right)\right]$$

$$=ab\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega+b^2\cos\omega}\right).$$

To find the area of $AFDG$ we have only to put $\pi-\omega$ in place of ω, and thus we get

$$ab\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega-b^2\cos\omega}\right).$$

$\therefore$ Area of $NCDM$

$$=2ab\left[\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega+b^2\cos\omega}\right)+\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega-b^2\cos\omega}\right)\right]$$

$$=2ab\tan^{-1}\left[\frac{2ab}{(a^2-b^2)\sin\omega}\right].$$

Also solved by *G. B. M. ZERR*.

115. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The axes of two right elliptic cylinders intersect at right angles in such a manner that the *major axes of the sections* are perpendicular. Supposing the axes to be $(A, B)>(a, b)$, what is the common volume?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/B^2+z^2/A^2=1$, $x^2/a^2+y^2/b^2=1$, be the equations to the cylinders.

$$\therefore V=\frac{8Ab}{aB}\int_0^B\sqrt{[(a^2-x^2)(B^2-x^2)]}dx,\ B<a\ldots\ (1).$$

$$V=\frac{8Ab}{aB}\int_0^a\sqrt{[(B^2-x^2)(a^2-x^2)]}dx,\ a<B\ldots.(2).$$

Let $x=B\sin\theta$ in (1).

Then $V=8ABb\int_0^{\frac{1}{2}\pi}\sqrt{[1-(B^2/a^2)\sin^2\theta]}\cos^2\theta d\theta$

$$=\frac{8Aa^2b}{3B}\{[1+B^2/a^2]E[B/a,\ \tfrac{1}{2}\pi]-[1-B^2/a^2]F[B/a,\ \tfrac{1}{2}\pi]\}.$$

Let $x=a\sin\theta$ in (2).

Then $V=8Aab\int_0^{\frac{1}{2}\pi}\sqrt{[1-(a^2/B^2)\sin^2\theta)\cos^2\theta d\theta}$

$$=\frac{8AB^2b}{3a}\{[1+a^2/B^2]E[a/B,\ \tfrac{1}{2}\pi]-[1-a^2/B^2]F[a/B,\ \tfrac{1}{2}\pi]\}.$$

MECHANICS.

121. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Two equal scale pans of mass m hang at rest over a smooth pulley. An inelastic particle, mass M, is dropped from a height h into one pan, and simultaneously another of equal mass and elasticity e is dropped from the same heignt into the other. Prove that every impact occurs when the pans are in their original positions, and find the total space described by either pan before motion ceases.

Soluiion by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The velocity of each particle just before impact$=\sqrt{(2gh)}$.

The velocity of first rebound of elastic particle$=e\sqrt{(2gh)}$.

The velocity of second rebound of elastic particle$=e^2\sqrt{(2gh)}$.

The velocity of nth rebound of elastic particle$=e^n\sqrt{(2gh)}$.

The elastic particle imparts a velocity to the scale pans at first impact $=\frac{M(1+e)}{2m+M}\sqrt{(2gh)}$, at the second impact a velocity$=\frac{Me(1+e)}{2m+M}\sqrt{(2gh)}$, at the nth impact a velocity$=\frac{Me^{n-1}(1+e)}{2m+M}\sqrt{(2gh)}$.

The inelastic particle imparts a velocity to the scale pans at first impact $=\frac{M}{2m+M}\sqrt{(2gh)}$.

Resultant velocity of these two particles at first impact$=\frac{Me}{2m+M}\sqrt{(2gh)}$.

The acceleration caused by inelastic particle$=\frac{Mg}{2m+M}$.

The time required for the scale pans to return to their original positions

$$=\frac{2Me}{2m+M}\sqrt{(2gh)}/\frac{Mg}{2m+M}=2e\sqrt{(2h/g)}.$$

The time required for the elastic particle to return to the same position $2e\sqrt{(2gh)}/g=2e\sqrt{(2h/g)}$.

Therefore, second impact takes place in original position.

As the scale pans have a velocity $=\frac{Me}{2m+M}\sqrt{(2gh)}$, the resultant velocity after the second impact

$$=\frac{Me(1+e)}{2m+M}\sqrt{(2gh)}-\frac{Me}{2m+M}\sqrt{(2gh)}=\frac{Me^2}{2m+M}\sqrt{(2gh)}.$$

The time required for the scale pans to return to their original position

$$=\frac{2Me^2}{2m+M}\sqrt{(2gh)}/\frac{Mg}{2m+M}=2e^2\sqrt{(2h/g)}.$$

It takes the elastic particle a time $=2e^2\sqrt{(2gh)}/g=2e^2\sqrt{(2h/g)}$.

Therefore, the third impact takes place in original position.

Proceeding thus, we find the resultant velocity after the nth impact $\frac{Me^n}{2m+M}\sqrt{(2gh)}$, and the time required for the scale pans to return to the original position $=2e^n\sqrt{(2h/g)}$.

Therefore, every impact occurs in the original position.

Space passed over by scale pans between nth and $(n+1)$th impact

$$\frac{Mg}{2m+M}\times\frac{2he^{2n}}{g}=\frac{2Mhe^{2n}}{2m+M}.$$

Therefore, total space passed over $=\frac{2Mh}{2m+M}(e^2+e^4+e^6+e^8+\ldots.)$

$\therefore S=\frac{2Mh}{2m+M}\cdot\frac{e^2}{1-e^2}$.

The elastic particle passes over a space S where

$$S=h+2he^2+2he^4+2he^6+\ldots.=\frac{1+e^2}{1-e^2}h.$$

The time required for scale pans to come to rest

$$=2\sqrt{\left(\frac{2h}{g}\right)}(e+e^2+e^3+e^4+\ldots.)=2\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{e}{1-e}.$$

The time required for the elastic particle to come to rest

$$=2\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{e}{1-e}+\sqrt{\left(\frac{2h}{g}\right)}=\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{1+e}{1-e}.$$

DIOPHANTINE ANALYSIS.

85. Proposed by A. H. BELL, Hillsboro, Ill.

Given $x^2-85\frac{1}{4}y^2=5$. What is the value of x and y in whole numbers?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$x^2-85\frac{1}{4}y^2=5$, $4x^2-341y^2=20$, $x^2=5+{}^{3}_{4}{}^{4}_{1}y^2$.

Let $y=2z$. $\therefore\ x^2=5+341z^2$.

Let $z=2$. Then $x=37$, $y=4$ are the least integral values.

86. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that $x^2+1457\equiv 0 \pmod{2389}$ is insoluble.

Solution by J. W. YOUNG, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y., and H. S. VANDIVER, Bala, Pa.

We are to prove that -1457 is a quadratic non-residue of the odd prime 2389. Using Legendre's symbol, we must show that

$$\left(\frac{-1457}{2389}\right)=-1.$$

We have $\left(\frac{-1457}{2387}\right)=\left(\frac{-1}{2389}\right)\left(\frac{31}{2389}\right)\left(\frac{47}{2389}\right)$.

$\left(\frac{-1}{2389}\right)=+1$, since 2389 is of the form $4n+1$.

By the law of reciprocity, we have $\left(\frac{31}{2389}\right)=\left(\frac{2389}{31}\right)=\left(\frac{2}{31}\right)=+1$, since in the first place 2389 is of the form $4n+1$, and since, secondly, 31 is of the form $8n\pm 1$.

$$\left(\frac{47}{2389}\right)=\left(\frac{2389}{47}\right)=\left(\frac{39}{47}\right)=-\left(\frac{47}{39}\right)=-\left(\frac{8}{39}\right)=-\left(\frac{2^2}{39}\right)\left(\frac{2}{39}\right)=-\left(\frac{2}{39}\right)=-1,$$

since 47 and 39 are both of the form $4n-1$, and 31 is of the form $2n\pm 1$.

Therefore, finally, $\left(\frac{-1457}{2389}\right)=(+1)(+1)(-1)=-1$.

Also solved by *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

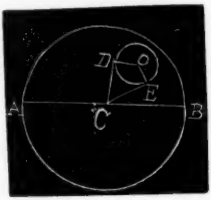
103. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

A circle is drawn at random both in magnitude and position, but so as to lie wholly on the surface of a given semi-circle. Show that the chance that a radius drawn at random in the semi-circle will cut the circle is

$$\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

Let C be the center of the semi-circle, O the center of the random circle, CD, CE tangents from C to random circle.

Also let $AC=r$, $CO=x$, $OD=z=OE$, $\angle ACO=\theta$, $\angle DCO=\varphi$, p=required chance.



The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $r/(1+\sin\theta)=x'$, and $r/(1+\sin\theta)=x'$ and r; of z, 0 and $x\sin\theta$, and 0 $r-x$.

Then $p=2\varphi/\pi$, $\varphi=\sin^{-1}(z/x)$.

$$\therefore\ p=\frac{\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\int_0^{r-x}\sin^{-1}(z/x)x\,dx\,dz+\int_0^{x'}\int_0^{x\sin\theta}\sin^{-1}(z/x)x\,dx\,dz\right]d\theta}{\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\int_0^{r-x}x\,dx\,dz+\int_0^{x'}\int_0^{x\sin\theta}x\,dx\,dz\right]d\theta}$$

$$=\frac{72}{\pi r^3(3\pi-4)}\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\int_0^{r-x}\sin^{-1}(z/x)x\,dx\,dz+\int_0^{x'}\int_0^{x\sin\theta}\sin^{-1}(z/x)x\,dx\,dz\right]d\theta$$

$$=\frac{72}{\pi r^3(3\pi-4)}\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\left[(r-x)\sin^{-1}\left(\frac{r-x}{x}\right)+\sqrt{(2rx-r^2)}-x\right]x\,dx\right.$$

$$\left.+\int_0^{x'}(\theta\sin\theta+\cos\theta-1)x^2\,dx\right]d\theta$$

$$=\frac{4}{\pi(3\pi-4)}\int_0^{\frac{1}{2}\pi}\left[2-\frac{3\theta}{(1+\sin\theta)^2}-\frac{(2+3\sin\theta+\sin^2\theta)\cos\theta}{(1+\sin\theta)^3}\right]d\theta$$

$$=\frac{4}{\pi(3\pi-4)}(\pi-\tfrac{1}{4}-2\log 2)=\frac{1}{3\pi-4}\left(4-\frac{1}{\pi}-\frac{8}{\pi}\log 2\right).$$

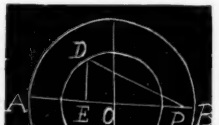
104. Proposed by **LON C. WALKER, A. M.**, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

In a given sphere two radii are drawn at random, and a point taken in each at random. (1) Find the chance that the distance between the two points does not exceed the radius of the sphere. (2) Find the distance between them.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be one of the points on radius OB, D the second point on radius OD. Let $OB=r$, $OP=x$, $OC=OD=y$, $OE=z$.

Then $PD=\sqrt{(x^2+y^2+2xz)}$.



The first point lies on the surface of a sphere radius x. An element of surface of this sphere is $2\pi x\,dx$. The second point lies on the surface of the sphere radius y. An element of surface of this sphere is $2\pi y\,dy$.

Let $(r^2-x^2-y^2)/2x=z'$, p=chance, and Δ=the average distance. Then

(1). $p=\frac{1}{(\frac{4}{3}\pi r^3)\pi r^2}\int_0^r\int_0^r\int_{-y}^{z}2\pi x dx.2\pi y dy dz=\frac{3}{2r^5}\int_0^r\int_0^r y[r^2-(x-y)^2]dxdy$

$$=\frac{1}{8r^5}\int_0^r(3r^4-6r^2x^2+8r^3x)dx=\frac{5}{8}.$$

(2). $J=\frac{1}{(\frac{4}{3}\pi r^3)\pi r^2}\int_0^r 2\pi x dx\Big[\int_0^x\int_{-y}^{y}2\pi y\sqrt{(x^2+y^2+2xz)}dydz$

$$+\int_x^r\int_{-y}^{y}2\pi y\sqrt{(x^2+y^2+2xz)}dydz\Big].$$

$$\therefore J=\frac{1}{r^5}\int_0^r dx\Big[\int_0^x y[(x+y)^3-(x-y)^3]dy+\int_z^r y[(y+x)^3-(y-x)^3]dy\Big]$$

$$=\frac{1}{10r^5}\int_0^r(15r^4x+10r^2x^3-x^4)dx=\frac{59}{60}r.$$

Also solved by the *PROPOSER*.

105. Proposed by L. C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find the average distance of the center of an ellipsoid, axes $2a$, $2b$, and $2c$ from its surface.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Different interpretations will result in different solutions. The method generally employed by the best thinkers in this department of mathematics is the following:

$d=\dfrac{\int r\,ds}{\int ds}$, where ds is an element of surface.

Let s be the eighth part of the surface of the ellipsoid.

Then $d=\frac{1}{s}\int r\,ds$.

$$ds=\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}dxdy=\left[\frac{a^4b^4-b^4(a^2-c^2)x^2-a^4(b^2-c^2)y^2}{a^2b^2(a^2b^2-b^2x^2-a^2y^2)}\right]^{\frac{1}{2}}dxdy$$

$$r=\sqrt{(x^2+y^2+z^2)}=\frac{1}{ab}[a^2b^2c^2+b^2(a^2-c^2)x^2+a^2(b^2-c^2)y^2]^{\frac{1}{2}}.$$

$$\therefore p=\int_0^a\int_0^{(b\,a)\sqrt{(a^2-x^2)}}[c^2+x^2+y^2-\frac{c^2}{b^4}(b^2-c^2)y^2-\frac{c^2}{a^4}(a^2-c^2)x^2$$

$$+\frac{c^2(x^2+y^2)(b^4x^2+a^4y^2)}{a^2b^2(a^2b^2-b^2x^2-a^2y^2)}]^{\frac{1}{2}}\,dxdy.$$

This expression does not seem to be easy to integrate.

MISCELLANEOUS.

96. Proposed by H. M. CASH, Lore City, Guernsey County, Ohio.

A stick of timber is 12 feet long, 8 inches deep, and 3 inches wide at one end, and 5 inches deep, and 12 inches wide at the other end. An what distance from either end should it be cut to divide it into two equal parts?

Solution by H. C. WHITAKER, A.M., Ph.D., Manual Training School, Philadelphia, Pa., and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

By the Prismoidal Formula, Volume$=\frac{1}{6}$(upper base+lower base+4 middle section)$\times$height.

Hence, the volume of the whole stick equals

$$\tfrac{1}{6}(8\times3+5\times12+4\times\tfrac{1}{2}^3\times\tfrac{1}{2}^5)\times144=6696.$$

Denote the distance of the required section from the 8×3 end by x, the depth at that section by y, the width by z. Then $y=8-\frac{1}{48}x$, and $z=3+\frac{1}{16}x$; also

$$\tfrac{1}{6}\left[8\times3+yz+4\left(\frac{8+y}{2}\right)\left(\frac{3+z}{2}\right)\right]x=3348.$$

Whence $x^3-504x^2-55296x+7713792=0$.

$x=84.8856$ inches$=7$ feet, 0.8856 inches.

Also solved by *R. L. MOORE*.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A spherical soap-bubble is electrified in such a way that the excess of the internal over the external air pressure is 2π when the bubble is in equilibrium. How does the tension of the film vary with the electric density?

Solution by the PROPOSER.

Let p be the excess of internal over external pressure, σ=electrical density, t=tension, r=radius of bubble.

Then $p+2\pi\sigma^2=2t/r$, (see Minchin, Vol. II, page 487, third edition).

But $p=2\pi$.

$\therefore 2\pi r(1+\sigma^2)=2t$, or $\dfrac{t}{1+\sigma^2}=\pi r$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

148. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

According to his contract a professor is to receive $1800. in cash plus board, etc., for his services during a scholastic year of nine months. This sum is payable in equal installments of $200. at the end of each scholastic month. The treasurer, however, paid the professor in ten equal installments of $180. The last two installments were paid Monday and Thursday of the last week in the scholastic year. Regarding money worth 6%, out of how much was the professor defrauded by the wiley treasurer?

149. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A wine cask contains 256 gallons of wine; a certain quantity is drawn off and the cask is filled with water; the same quantity of the mixture is drawn off and the cask again filled with water and so on for four draughts, when there remain only 81 gallons of wine in the cask. How many gallons of wine are drawn at each of the draughts?—[*Colburn's Algebra.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

ALGEBRA.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Factorize $2b^2c^2+2c^2a^2+2a^2b^2+2a^2d^2+2b^2d^2+2c^2d^2-a^4-b^4-c^4-d^4$.

146. Proposed by B. F. YANNEY, Professor of Mathematics, Mount Union College, Alliance, Ohio.

If the series 1, 3, 5, ... $2n-1$, ... be divided into successive groups of r terms each, the sum of the terms of the nth group will be $(2n-1)$ times the sum of the terms of the first group, or $(2n-1)r^2$.

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

GEOMETRY.

175. Proposed by W. P. WEBBER, Mississippi Normal College, Houston, Miss.

A field is enclosed by a fence in circular form and a straight gate 20 feet wide. The fence is 100 feet in length. How much land in the field? [Solution by most elementary method possible.]

176. Proposed by R. A. WELLS, Franklin College, New Athens, Ohio.

If there be three straight lines which meet in a point, and the arbitrary constants of their equations, expressed in the slope form, be taken as the coördinates of three points, these three points will lie in a straight line.

*** Solutions of these problem should be sent to B. F. Finkel not later than Dec. 10.

CALCULUS.

116. Proposed by M. E. GRABER, A. B., Tutor in Mathematics, Heidelberg University, Tiffin, Ohio.

Find the curve the length of whose arc measured from a given point is a mean proportional between the ordinate and twice the abscissa.

117. Proposed by WM. FRED FLEMING, Chicago, Ill.

A tin watering-pot is constructed by joining the frustums of two right cones, so that their intersection is a mathematical one, their axes meeting at an angle of 45°. The bases of the smaller frustum are 2 inches and 4 inches in diameter, its altitude 8 inches. The bases of the larger frustum are 10 inches and 12 inches in diameter, its altitude 15 inches. In joining the two frustums the edges of the two larger bases are brought into coincidence. Water is poured into the vessel until it begins to run out of the spout. How many gallons (231 cubic inches) are required? How much water is in the spout and how much in the can? The vessel is tilted forward (in the plane of the axes of the two frustums) sufficiently to allow one-half of the water to run out. How much of the liquid is left in the spout and can, and what is the area of the surface of the water in spout and can? Through what angle has the vessel been tilted?

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

MECHANICS.

126. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, Stroud, London, England.

AB is the horizontal base of a smooth cycloidal tube, vertex downward. A sphere is placed in the tube at A, and when it reaches the vertex another sphere of different mass is placed in the tube at B. When and where do they meet, and find their velocity immediately after collision, the spheres being partially elastic.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Develop the Fourier Series to represent the temperature of a circular wire of uniform cross-section, in which the temperatures of the four quadrants are in order $t, 2t, 3t, 4t$.

128. Proposed by M. E. GRABER, A. B., Heidelberg University, Tiffin, Ohio.

A particle is placed on the convex side of a smooth ellipse and is acted upon by two forces, F and F', towards the foci, and a force, F'', towards the center. Find the position of equilibrium.

129. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Two spheres whose masses are M_1 and M_2 are a units apart, and attract each other with a force $= M_1 M_2 / a^2$. Find work done in carrying a unit mass from the center point between them a distance r in a direction θ with line of centers.

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

DIOPHANTINE ANALYSIS.

91 Proposed by L. C. WALKER, A.M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find the three least positive integral numbers whose sum, sum of their squares, and sum of their cubes shall each be rational squares.

92. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

AVERAGE AND PROBABILITY.

115. Proposed by L.C.WALKER, A.M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Three points are at random within a given triangle. Find the chance that they will all lie on one side of some one line that can be drawn through the center of gravity of the triangle.

116. Proposed by the late ENOCH BEERY SEITZ,

The average area of the quadrilateral formed by joining four radom points on the surface of a circle, radius a, is $\frac{4a^2}{3\pi}$.

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

MISCELLANEOUS.

117. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

If $x\cos\alpha+y\cos\alpha=a\cos\theta+b\cos\varphi$, and $x\sin\alpha+b\sin\varphi=y\sin\alpha+a\sin\theta=\kappa$, find the maximum value of κ, and the values of x and y.

118. Proposed by O. W. ANTHONY, New York, N. Y.

If f is determined by the equation $f(\mu\nu)=f(\mu)f^{-1}(\nu)+f(\nu)f^{-1}(\mu)$, when f^{-1} is the inverse of f, show that $f[(2)^\mu]=\frac{k^{\mu+1}}{2^{\mu+1}}$, where k is the constant.

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

BOOKS AND PERIODICALS.

Annals of Mathematics. Published under the auspices of Harvard University. Second series, Vol. 2, No. 4. Price, $2.00 per year in advance. Published in October, January, April, and July.

The July number of the current year contains the following articles: Concerning du Bois-Reymond's Two Relative Integrability Theorems, by Professor E. H. Moore; On a Theorem of Kinematics, by Dr. P. Saurel; The Collineations of Space whichTrans forms a Non-Degenerate Quadratic Surface into Itself, by Dr. R. G. Wood; Note on Multiply Perfect Numbers, by Dr. J. Westlund; The Isoperimetrical Problem on Any Surface, by Mr. J. K. Whitmore; On a Surface of the Sixth Order which is Touched by the Axes of all Screws Reciprocal to Three Given Screws, by Professor E. W. Hyde; Note Sur l' évalution d'une intégral define, Par le Professor D. Sintsof. B. F. F.

The American Journal of Mathematics. Published under the auspices of the Johns Hopkins University and edited by Frank Morely with the coöperation of other mathematicians. Price, $5.00 per year in advance.

The October number contains the following articles: Memoir on the Algebra of

Symbolic Logic, by A. N. Whitehead; Secular Perterbations of the Planets, by G. W. Hill; Representation of Linear Groups as Transitive Groups, by Leonard E. Dickson; A Class of Number Systems in Six Units, by G. P. Starkweather. B. F. F.

Éléments de Mathématiqes Supérieur, A l'Usage des Physiciens, Chimistes et Ingénieurs et des Aspirants a ces Titres, des Conducteurs de Travaux, et des Élèves des Facultés des Scienceset des Écoles Industrielles. Par B. Vogt, Ancien Élève de l'École Normale Supérieur Agrégé de Sciences Mathématiques, Docteur de Sciences, Professeur a l'Université de Nancy. Paris, France, Librairie Nony te C^ie^.

This work is divided into seven parts. The first part treats on Algebra; the second part on Analytical Geometry; the third part on Derivatives and Differentials; the fourth part on Theory of Equations; the fifth part on Geometric Applications; sixth part, Integral Calculus; seventh part, Differential Equations. Pages 485-565 are devoted to notes on Algebra and Analytic Geometry, in which are treated Series, Elimination, Transformation of Coördinates, Curves on Surfaces, etc. Pages 569-603 are devoted to exercises in the various subjects of which the book treats. We are sure that this work will be of great value to the practical mathematician as well as to the student of general mathematics.

B. F. F.

Linear Groups, With an Exposition of the Galois Field Theory. By Leonard Eugene Dickson, Ph. D., Assistant Professor of Mathematics in the University of Chicago. Large 8vo, cloth, 312 pages. Leipzig, Germany: B. G. Teubner.

This volumn is intended as an introduction to the subject of Linear Groups in a Finite Field, a subject having immediate application in many problems of Geometry and Function Theory. The earlier chapters of the book are devoted to an elementary exposition of the theory of Galois Fields chiefly in their abstract form, while the second part of the book is devoted to an elementary exposition of the more important results concerning Linear Groups in a Galois Field.

Dr.Dickson is the first American mathematician who has written a book on the Theory of Groups, and in this work he has added a valuable contribution to the literature of the subject. B. F. F.

Geometric Exercises in Paper Folding. By T. Sundara Row. Edited and revised by W. W. Beman and D. E. Smith. With many half-tone engravings from photographs of actual exercises and a package of colored papers for folding. 8vo, cloth, x+148 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

This book contains proofs of a number of geometric theorems by means of folding paper. By this means some important geometric processes can be effected much more easily than with compasses and straight edge. While it is not possible to fold paper so as to represent a circle, yet a number of points on a circle can be obtained by this method. In this book are treated, The Square, the Equilateral Triangle, Squares and Rectangles. Pentagon, Hexagon, the Octagon, the Nonagon, the Decagon. and the Dodecagon, th Pentedecagon, Series, Polygons, General Principles, the Conic Sections, the Circle, the Ellipse, the Parabola, the Hyperbola, Miscellaneous Curves, the Cissoid, the Conchoid, the the Witch, the Cubical Parabola, the Harmonic Curve, the Ovals of Cassina, the Logarithmic Curve, the Catenary, the Cardioid, the Limaçon, the Lemniscate of Bernouilli, and the Cycloid. The work is one by which new interest may be awakened in the ever interesting and fascinating subject of Geometry. B. F. F.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. VIII. NOVEMBER, 1901. No. 11.

ON THE HISTORY OF SEVERAL FUNDAMENTAL THEOREMS IN THE THEORY OF GROUPS OF FINITE ORDER.*

By DR. G. A. MILLER, Leland Stanford Jr. University, Palo Alto, Cal.

The three most prominent sources of the theory of groups of finite order are: geometric transformations, theory of numbers, and theory of algebraic equations. The group properties of the totality of the rotations through a submultiple of 360°, or through 180° around three concurrent lines, each perpendicular to the plane determined by the other two, must have been observed very early. The prominent place which the five regular polyhedra—the Platonic bodies—occupy in the history of thought make it appear probable that the group properties of the totality of the rotations into themselves of these solids were observed quite early.†

The elementary theory of congruences and especially the combinatory laws of the n roots of unity when they are multiplied together exhibit group properties which cannot have escaped the notice of the early students of these questions. As Cayley aptly says (*Philosophical Magazine*, 1854, page 40) "A number of elementary group concepts have been employed by mathematicians for a long time but have not been especially noted on account of their great simplicity." Some of the theorems by Gauss and Schering along this line are of sufficient interest in themselves to attract attention. In particular the theorem by Gauss to the effect that an Abelian group can be resolved in only one way in-

*Read before the American Association for the Advancement of Science, August, 1901.

†Bravais, *Liouville's Journal*, Vol. 14, 1849, page 167.

to factor groups whose orders are prime to each other, and the theorem by Schering to the effect that every Abelian group can be resolved in more than one way into factor groups such that the order of each is divisible by the orders of all those which follow it, have a strong claim to being considered the beginning of the theory of abstract groups of finite order.

The early developments in substitution groups were made in connection with the theory of algebraic equations, starting with the theories of Lagrange and becoming prominent in the development and extension of these theories by Ruffini, Abel, and Galois.* Cauchy seems to have been the first to develop the theory of substitution groups independently of any direct application to the theory of equations, and he was also the first to prove any remarkable theorems in this theory. The earlier theorems were almost self-evident, but this cannot be said of the theorem that a group must involve a substitution of a prime order (p) whenever its order is divisible by p.† This theorem, known in the theory of groups as Cauchy's theorem, is especially interesting since it is independent of the particular notation by means of which a group is represented and hence applies directly to the theory of abstract groups.

In proving this theorem, Cauchy employed another interesting theorem, due to himself, viz., that the symmetric group of degree n contains a subgroup of order p^m, p^m being the highest power of p that divides $n!$ About thirty years later Sylow extended very materially these results due to Cauchy by proving that a group (G) of order g must always contain $1+kp$ subgroups of order p^α, p^α being the highest power of p that divides g, and that all of these subgroups are conjugate under G. This fundamental theorem is known as *Sylow's theorem*. It is sometimes called the Cauchy-Sylow theorem, and was first published in the *Mathematische Annalen*, Vol. 5.

About twenty years later Frobenius published, in the *Berliner Sitzungsberichte*, a very important extension of Sylow's theorem in which it is proved that the number of subgroups of order p^α which are contained in any group is $\equiv 1 \bmod p$. These subgroups do not necessarily form a single set of conjugates unless p^α is the highest power of p that is contained in the order of the group. Frobenius deserves credit also for a simple proof of Sylow's theorem by means of a more prominent use of the concept of complete sets of conjugates. This concept is prominent in a large number of the publications by Frobenius.

From what precedes it may be observed that a very important part of Sylow's theorem was constructed on French soil. A Norwegian greatly enlarged and beautified this structure. Finally a German enhanced its usefulness by extending it very materially in certain directions. Hence this fundamental theorem stands before us as a truly international structure, which has required more than fifty years for its erection. One of its main contributors (Sylow) has published comparatively little along the line of group theory, while the other two have published extensively along this line.

---

*Cauchy, *Physique Mathematique*, Vol. 3, 1844, page 250.

†Pierpont, *Bulletin of the American Mathematical Society*, Vol. 1, 1895, page 198.